

On semantic cutting planes proofs with very small coefficients

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Abstract

Cutting planes proofs for integer programs can naturally be defined both in a syntactic and in a semantic fashion. Filmus et al. (STACS 2016) proved that semantic cutting planes proofs may be exponentially stronger than syntactic ones, even if they use the semantic rule only once. We show that when semantic cutting planes proofs are restricted to have coefficients bounded by a function growing slowly enough, syntactic cutting planes can simulate them efficiently. Furthermore if we strengthen the restriction to a constant bound, then the simulating syntactic proof even has polynomially small coefficients.

Keywords: Proof Complexity, Cutting Planes, Semantic Proofs

1. Introduction

The field of *proof complexity* studies the length of proofs for propositional unsatisfiability, also called refutations. The historical motivation was the P vs NP problem. If there are unsatisfiable formulas without short refutations, then it must be that NP is different from co-NP, and therefore that P is different from NP [CR79]. In this context a proof must be efficiently verifiable and therefore written in some clear format, in some specific *proof system*. If this format is simple enough, we can sometimes show strong lower bounds on the length of such proofs. As in circuit complexity, proving lower bounds is hard even for some apparently simple proof systems.

There are other good reasons to study proof systems. Algorithms which solve unsatisfiability implicitly produce refutations in a relatively simple proof system. See for example the well known connection between DPLL algorithms, decision trees and treelike resolution proofs [DP60, DLL62, BSIW04, BGL13]. Another classic example, more relevant for this paper, is the use of Gomory cuts to solve integer programs [Gom58]. Algorithms that mix branch and bound techniques, linear programming and Gomory cuts can be often formalized as proofs in the *cutting planes* proof system [Chv73, CCH89].

Despite the importance of the system, the only method we know to lower-bound the length of cutting planes proofs is *interpolation* [Kra97], which was used to prove the first lower bounds [Pud97]. Recently a variant of this method has been applied to random k -CNFs, with $k = \omega(1)$, as well [FPPR17, HP17].

Most systems studied in proof complexity, including cutting planes, are actually inference systems. A proof is developed line by line, and each line is either an axiom of the system or is derived from some previous lines according to a specific inference rule. Nevertheless it turns out that the specifics of the inference rules are not important for many results in the area, and the main factor in the power of the proof system is the expressivity of the proof lines. Thus it makes sense to study both *syntactic* and *semantic* proofs. In the former a specific set of inference rules are available to derive a new proof line from lines derived before. In the latter a new line can be an arbitrary logical consequence of a constant number of previously derived lines.¹

A similar (but more powerful) form of semantic proofs naturally occur in the study of proof space [ET01, ABSRW02]. In this framework a proof is seen as a sequence of memory configurations, each consisting of a set of proof lines, and each configuration is semantically implied by the previous one. This approach can be used to study the memory usage of proof verification algorithms. Most successful lower-bound techniques related to proof space, either based on connection to other complexity measures [AD08, FLM⁺14], to pebbling games [Nor13, GLT80, CLNV15], or to matching games [FLN⁺15, BSG03, BG14, FLM⁺13], work against this type of semantic proofs. Only limited results are known for the space complexity of CP proofs, though [PGT15].

If we study proof length, the appropriate semantic version of cutting planes is the one that infer any new inequality ℓ which logically follows (over $\{0, 1\}^n$) from two previously derived inequalities. Observe that this is not a proof system in the technical sense, because there is no known efficient algorithm to verify whether an inference step is sound. Indeed, even to check whether the two linear inequalities $\sum_i a_i x_i \leq b$ and $\sum_i a_i x_i \geq b$ are simultaneously satisfiable over $x_i \in \{0, 1\}$ is NP-complete if the coefficients have exponential magnitude with respect to the number of variables (it is the Subset Sum problem). The situation is different with small coefficients – see the discussion at the end of this note.

Semantic cutting planes seems to be a much stronger proof system than syntactic, and indeed even allowing just one application of the semantic rule (together with the usual syntactic rules) gives an exponential advantage over purely syntactic cutting planes [FHL16]. Still, the same paper shows that the formula that [Pud97] proved to be hard for syntactic CP is hard for semantic CP as well. If semantic CP is stronger in general, is there any condition under which syntactic CP efficiently simulates semantic CP? In this paper we show that

Theorem 1 (Informal). *A semantic cutting planes proof in which all coefficients have very small size can be transformed into a syntactic cutting planes proof with at most a polynomial blowup in size. If the coefficients in the semantic proof are constant, the coefficients in the syntactic proof can be made polynomial.*

The idea of the proof is to realize that if the coefficients have small size, then the linear inequalities involved in the inference must have a lot of symmetries, hence the argument can be viewed as proving the soundness of an inference rule with a small number of variables. The main contribution of this paper is to show that this can be

¹The limitation to a constant number of premises keeps the proof systems from being trivial.

done in syntactic CP. Compare this result with the separation in [FHL16]. They exhibit a short semantic CP refutation for a CNF which is hard for syntactic CP. Such a refutation uses exponential magnitude coefficients.

The paper is organized as follows. In Section 2 we give the necessary definitions and notation. In Section 3 we discuss implicational completeness of CP and prove some upper bounds. Finally in Section 4 we show our main result, namely that semantic proofs with very small coefficients can be simulated by syntactic proofs. We conclude the paper with some open problems.

2. Preliminaries

We consider *cutting planes (CP)* [Chv73, CCT87], a proof system based on manipulation of inequalities over variables x_1, \dots, x_n . Each line in the proof is an inequality of the form $\sum_i a_i x_i \geq b$ where $a_i, b \in \mathbb{Z}$. Variables x_1, \dots, x_n are understood to take integer values.

A *syntactic CP derivation* of an inequality ℓ_τ from a set of inequalities \mathcal{S} is a sequence of inequalities $(\ell_1, \dots, \ell_\tau)$ such that for $1 \leq i \leq \tau$ the inequality ℓ_i is either in \mathcal{S} or is obtained by one of the following rules.

- **Sum:** We can add two earlier inequalities.
- **Multiplication:** We can multiply an inequality by a positive integer.
- **Division:** From an inequality $\sum_i a_i x_i \geq b$ we can derive

$$\sum_i (a_i/c) x_i \geq \lceil b/c \rceil$$

if c is a positive integer which divides all coefficients a_i .

When used as a propositional proof system (which is not how we will be using it in the next section), a syntactic CP derivations may also include

- **Boolean axioms:** We can introduce inequalities $x_i \geq 0$ and $-x_i \geq -1$.

A *semantic CP derivation* of an inequality ℓ_τ from a set of inequalities \mathcal{S} is a sequence of inequalities $(\ell_1, \dots, \ell_\tau)$ such that for $1 \leq i \leq \tau$ the inequality ℓ_i is either in \mathcal{S} or follows semantically from two earlier inequalities ℓ_j and ℓ_k , in the sense that ℓ_i holds for any point in $\{0, 1\}^n$ where ℓ_j and ℓ_k both hold. We will also consider semantic entailment over \mathbb{Z}^n rather than $\{0, 1\}^n$, but we do not need a formal definition of derivations of this kind.

A syntactic (resp. semantic) CP refutation of \mathcal{S} is a syntactic (resp. semantic) CP derivation of $0 \geq 1$.

If we do not care to specify the coefficients, we abbreviate $\sum_i a_i x_i \geq b$ as $A\bar{x} \geq b$. For our convenience we sometimes write $A\bar{x} \leq b$ as an alias for $-A\bar{x} \geq -b$ and $A\bar{x} = b$ as a shorthand for the conjunction of the inequalities $A\bar{x} \geq b$ and $A\bar{x} \leq b$. The *length* of a CP derivation is the number of steps. The *magnitude* of a CP derivation is the maximum absolute value among the coefficients and constants in all its inequalities.

The *size* of a CP derivation is the sum, over all inequalities, of the binary length of all coefficients and the constant of each inequality. Clearly the size is at most polynomial in the length times \log_2 of the magnitude.

Cutting planes never needs coefficients or constants of magnitude more than exponential in the proof length, hence proofs of polynomial length can be made of polynomial size [CCT87]. Restricting to polynomial magnitude, which is the goal of one of our simulations, gives a robust, natural system CP* [BPR97]. It can be thought of as cutting planes in which all constants and coefficients are written in unary. The system restricted to coefficients in the set $\{-2, -1, 0, 1, 2\}$ is already exponentially stronger than resolution, as it has short refutations of the pigeonhole principle [CCT87].

3. Completeness

To prove our main result in the next section we need to show, with good bounds, that syntactic CP is implicationally complete, in the sense that every inequality ℓ that follows from \mathcal{S} over the integers is provable in a finite number of steps.

Cutting planes was originally introduced as a complete implicational system in [Chv73], but no quantitative bound was given there on the number of steps, nor on the magnitude of the coefficients involved. The refutational version of cutting planes was introduced in [CCT87], partly because, if you only consider refutations, there is a general upper bound on proof length in terms of dimension – see Theorem 4 below.

In our setting, we consider implications from systems of axioms which explicitly include upper and lower bounds on the values of all variables. Here it is straightforward to get useful upper bounds on implicational completeness by using results from [CCT87]. We also prove a version of implicational completeness from scratch to get the bounds we want on the magnitude of coefficients (which almost, but not quite, follow from [CCT87]).

We start with a lemma that allows us, under some conditions, to get rid of a hypothesis by paying for it with a weaker conclusion.

Lemma 2. *Let \mathcal{S} be a system of linear inequalities on n variables. Suppose \mathcal{S} contains an axiom $A\bar{x} \geq b$, and also that there is an integer bound γ such that \mathcal{S} contains axioms $0 \leq x_i \leq \gamma$ for every variable x_i . Then given a derivation*

$$\mathcal{S} \cup \{A\bar{x} \leq b\} \vdash C\bar{x} \geq d$$

of length τ and magnitude μ , we can find an integer $K \geq 0$ and a derivation

$$\mathcal{S} \vdash K(A\bar{x} - b) + C\bar{x} \geq d \tag{1}$$

either of length $O(\tau)$, or simultaneously of length $\tau \cdot \text{poly}(\mu, n, \gamma)$ and magnitude $\text{poly}(\mu, n, \gamma)$.

Proof. Let ℓ_1, \dots, ℓ_τ be the the proof of $C\bar{x} \geq d$. We will inductively define lines $\ell_1^*, \dots, \ell_\tau^*$ of the form required by the lemma, and show that each ℓ_i^* can be derived from $\ell_1^*, \dots, \ell_{i-1}^*$. Let us first ignore the bound on the magnitude, and just focus on building a proof of length $O(\tau)$. We distinguish various cases.

1. **Axiom:** If ℓ_i is the axiom $-A\bar{x} \geq -b$ (that is, $A\bar{x} \leq b$), we put $K = 1$, so that ℓ_i^* is the line

$$\ell_i^* : (A\bar{x} - b) - A\bar{x} \geq -b$$

which is just another way of writing $0 \geq 0$. Other axioms are unchanged.

2. **Sum, multiplication:** These are immediate from the definition.
 3. **Division:** ℓ_i is derived from ℓ_j for some $j < i$ by division. So we have

$$\begin{aligned} \ell_j &: gE\bar{x} \geq f \\ \ell_i &: E\bar{x} \geq \lceil f/g \rceil . \end{aligned}$$

By the inductive hypothesis, for some integer $K \geq 0$ we have derived

$$\ell_j^* : K(A\bar{x} - b) + gE\bar{x} \geq f .$$

We multiply the axiom $A\bar{x} \geq b$ by $g - 1$ to derive $(g - 1)(A\bar{x} - b) \geq 0$ and add this to ℓ_j^* to get

$$gK(A\bar{x} - b) + gE\bar{x} \geq f .$$

Dividing by g now gives an inequality ℓ_i^* of the right form.

The construction so far gives a derivation of length $O(\tau)$ with no guarantee on the magnitude. To keep the magnitude small it is sufficient just to bound coefficient during the construction of the lines ℓ_i^* for $1 \leq i \leq \tau$, since the magnitude of intermediate lines are polynomially related. We proceed by induction. Consider the i -th line $E\bar{x} \geq f$ in the original proof. We derive

$$K(A\bar{x} - b) + E\bar{x} \geq f \tag{2}$$

using one of the schemes 1, 2 and 3 explained above. This derivation involves up to two lines among $\{\ell_j^*\}_{j < i}$, which have coefficients bounded by $\text{poly}(n, \mu, \gamma)$ by inductive hypothesis. The inequalities $E\bar{x} \geq f$ and $A\bar{x} \geq b$ appear in the original proof, so it is enough to bound K , which by construction is already at most $\text{poly}(n, \mu, \gamma)$. Let $-\delta$ be the minimum value which can be taken by $E\bar{x}$ for $0 \leq \bar{x} \leq \gamma$. We derive $E\bar{x} \geq -\delta$ from \mathcal{S} in $2n - 1$ steps, by adding together suitably scaled versions of the variable bounding axioms. We can assume $f > -\delta$ as otherwise we could now set $K = 0$. Note that $|\delta| \leq n\mu\gamma$.

We will reduce K in (2) step by step until it reaches size $f + \delta$, and will set ℓ_i^* to be a resulting line. So suppose $K > f + \delta$. We first multiply (2) by $K - 1$ and add $E\bar{x} \geq -\delta$ to it, giving

$$(K - 1)K(A\bar{x} - b) + KE\bar{x} \geq Kf - f - \delta. \tag{3}$$

Then, since $f + \delta < K$, by dividing by K and rounding up we get

$$(K - 1)(A\bar{x} - b) + E\bar{x} \geq f.$$

We repeat this step until we get K down to $f + \delta$, and we set the resulting line to be ℓ_i^* . Such a derivation has length at most $\text{poly}(n, \mu, \gamma)$. \square

Lemma 3. *Let \mathcal{S} be a system of linear inequalities on n variables. Suppose \mathcal{S} contains axioms $0 \leq x_i \leq \gamma$ for every variable x_i , and that*

$$\mathcal{S} \cup \{A\bar{x} \leq b, A\bar{x} \geq b\} \vdash 0 \geq 1$$

in length τ and magnitude μ . Then

$$\mathcal{S} \cup \{A\bar{x} \geq b\} \vdash A\bar{x} \geq b + 1 \quad (4)$$

either in length $O(\tau)$, or simultaneously in length $\tau \cdot \text{poly}(\mu, n, \gamma)$ and magnitude $\text{poly}(\mu, n, \gamma)$.

Proof. We use $\mathcal{S} \cup \{A\bar{x} \geq b\}$ as the system \mathcal{S} in Lemma 2. This gives

$$\mathcal{S} \cup \{A\bar{x} \geq b\} \vdash K(A\bar{x} - b) + 0 \geq 1$$

for some integer $K \geq 0$. If $K = 0$ then we can get (4) by summing $A\bar{x} \geq b$ and $0 \geq 1$. Otherwise with one more division step we get

$$\mathcal{S} \cup \{A\bar{x} \geq b\} \vdash A\bar{x} \geq b + \lceil 1/K \rceil = b + 1.$$

Length and magnitude depend on which derivation we use from Lemma 2. □

We will now state two versions of quantitative refutational completeness for CP, which we will then use to give bounds on implicational completeness.

Theorem 4. *Let \mathcal{S} be a set of linear inequalities over n variables with no integral solution. There exists a syntactic CP refutation of \mathcal{S} of length $O(n^{3n+1})$.*

Proof. Theorem 1' of [CCT87], together with the remarks following it, gives a bound of n^{3n} on the number of lines in a refutation in which each line is obtained by taking a positive linear combination of earlier lines and rounding up. By Carathéodory's theorem, we may assume that each linear combination uses no more than $n + 2$ previous lines. □

Cook et al. [CCT87] also shows bounds on the magnitude of such a refutation, namely that the binary size of all coefficients and constant terms is polynomial in the binary size of \mathcal{S} . Our bound on magnitude below is slightly better when n is constant, as it will be in our applications, while our size bound is worse.

Theorem 5. *Let \mathcal{S} be a set of linear inequalities over n variables of magnitude μ , with no integral solution, and that contains $0 \leq x_i \leq \gamma$ for every variable x_i . When n is a constant, \mathcal{S} has a CP refutation of length and magnitude polynomial in μ and γ .*

Proof. We will use the notation \vdash_{μ}^{τ} to indicate a derivation of length τ and magnitude μ . Consider any tuple $\bar{a} \in \mathbb{Z}^n$ with $0 \leq \bar{a} \leq \gamma$. Since \mathcal{S} is unsatisfiable, it contains some inequality $B\bar{x} \geq c$ such that $B\bar{a} = d < c$. We first construct a derivation

$$\{x_i \leq a_i, x_i \geq a_i\}_{i=1}^n \vdash_{n\mu\gamma}^{2n-1} -B\bar{x} \geq -d.$$

We then add $B\bar{x} \geq c$ and divide by $c - d$, to get

$$\mathcal{S} \cup \{x_i = a_i\}_{i=1}^n \vdash_{n\mu\gamma}^{2n+1} 0 \geq 1$$

where $x_i = a_i$ is just shorthand for $x_i \leq a_i, x_i \geq a_i$.

Now fix any $\bar{a} \in Z^{n-1}$ with $0 \leq \bar{a} \leq \gamma$. For each $0 \leq b \leq \gamma$ we have

$$\mathcal{S} \cup \{x_i = a_i\}_{i=1}^{n-1} \cup \{x_n = b\} \vdash_{n\mu\gamma}^{2n+1} 0 \geq 1$$

so by Lemma 3 we have

$$\mathcal{S} \cup \{x_i = a_i\}_{i=1}^{n-1} \cup \{x_n \geq b\} \vdash_{\text{poly}(\mu, \gamma)}^{\text{poly}(\mu, \gamma)} x_n \geq b + 1.$$

Stringing these derivations together, and using the axioms $0 \leq x_n \leq \gamma$, we get

$$\mathcal{S} \cup \{x_i = a_i\}_{i=1}^{n-1} \vdash_{\text{poly}(\mu, \gamma)}^{\text{poly}(\mu, \gamma)} 0 \geq 1,$$

where we have multiplied the length by $\gamma + 1$, and not increased the magnitude. The theorem follows by repeating this step for x_{n-1}, \dots, x_1 . \square

We are now ready to prove our bounds on implicational completeness.

Theorem 6. *Let \mathcal{S} be a set of linear inequalities over n variables, which contains $0 \leq x_i \leq \gamma$ for each variable x_i . Suppose that \mathcal{S} entails $C\bar{x} \geq d$ over the integers and that μ is a bound on the magnitude of $\mathcal{S} \cup \{C\bar{x} \geq d\}$.*

Then there is a derivation $\mathcal{S} \vdash C\bar{x} \geq d$ of length $O(n^{3n+2}\mu\gamma)$, and also a derivation simultaneously of length and magnitude $\text{poly}(\mu, \gamma)$ (if we treat n as constant).

Proof. As in the proof of Lemma 2, for some integer δ with $|\delta| \leq n\mu\gamma$ there is a derivation $C\bar{x} \geq \delta$ of length $O(n)$ and magnitude $\leq n\mu\gamma$. If $\delta > d$ then we are done, since we can derive $0 \geq -1$ from the axioms bounding the variables.

Otherwise, observe that for $a = \delta, \dots, d - 1$ the set $\mathcal{S} \cup \{C\bar{x} = a\}$ is unsatisfiable over the integers. We imitate the construction in the proof of Theorem 5. For the bound just on length, by Theorem 4 for each a we have

$$\mathcal{S} \cup \{C\bar{x} = a\} \vdash 0 \geq 1$$

in length $O(n^{3n+1})$. Thus by Lemma 3

$$\mathcal{S} \cup \{C\bar{x} \geq a\} \vdash C\bar{x} \geq a + 1$$

also in length $O(n^{3n+1})$. Starting with $C\bar{x} \geq \delta$, and then using these $d - \delta$ derivations in series, gives the theorem (as necessarily $|d| \leq \mu$).

The simultaneous bound on length and magnitude follows by a similar argument, but using Theorem 5 instead of Theorem 4. \square

4. Semantic and syntactic CP

In this section we prove the main result of the paper, namely the simulation of semantic CP with small coefficients by syntactic CP. Let ℓ_1, \dots, ℓ_ν and $\hat{\ell}$ be inequalities such that ℓ_1, \dots, ℓ_ν semantically entail $\hat{\ell}$ over 0/1 assignments to the variables x_1, \dots, x_n . Suppose that the variables can be partitioned into sets B_1, \dots, B_m such that each inequality in $\ell_1, \dots, \ell_\nu, \hat{\ell}$ can be written in the form

$$a_1 \sum_{i \in B_1} x_i + \dots + a_m \sum_{i \in B_m} x_i \geq b. \quad (5)$$

In other words, variables x_i and x_j are in the same set B_k if, in each inequality $\ell_1, \dots, \ell_\nu, \hat{\ell}$, variable x_i has the same coefficient as x_j . Let $\ell'_1, \dots, \ell'_\nu, \hat{\ell}'$ be the result of writing all the inequalities in the above form, and then for each B_j replacing $\sum_{i \in B_j} x_i$ with a single variable y_j . For example, Equation (5) becomes

$$a_1 y_1 + \dots + a_m y_m \geq b. \quad (6)$$

Let T be the set of inequalities $\{0 \leq y_j \leq |B_j| : 1 \leq j \leq m\}$. Then, over the integers,

$$\{\ell'_1, \dots, \ell'_\nu\} \cup T \models \hat{\ell}'.$$

This is because any assignment to y variables satisfying the left hand side can be made into a 0/1 assignment to x variables satisfying ℓ_1, \dots, ℓ_ν ; hence $\hat{\ell}$ is true with this assignment to x variables, and so $\hat{\ell}'$ is true with the assignment to y variables.

If the number m of y variables is much smaller than n , we can build a relatively efficient syntactic CP derivation

$$\{\ell'_1, \dots, \ell'_\nu\} \cup T \vdash \hat{\ell}' \quad (7)$$

using the completeness results in the Section 3. Now take this derivation, and substitute each variable y_j with the corresponding sum $\sum_{i \in B_j} x_i$. Every inequality in T , after the substitution, has a short derivation from the boolean axioms for the x variables. Hence

$$\{\ell_1, \dots, \ell_\nu\} \vdash \hat{\ell}$$

with essentially no increase in size of magnitude with respect to the derivation in (7). We get the next two theorems by applying the argument above to each inference step of the semantic CP derivation. The results differ because of the strategy we use to get derivation (7).

Theorem 7 (Very small coefficients). *Any semantic cutting planes proof of magnitude $\sigma = O\left(\sqrt[3]{\frac{\log n}{\log \log n}}\right)$ is polynomially simulated by a syntactic cutting planes proof.*

Proof. Consider an inference step in the refutation. Semantic CP has binary rules, so three inequalities appear and for each variable x_i , the sequence of coefficients in front of x_i is one among $(2\sigma+1)^3$ possibilities. Therefore the n variables can be divided into no more than $(2\sigma+1)^3$ blocks, and the formula can be considered as if it had $m = (2\sigma+1)^3$ variables. We use Theorem 6 with m variables bounded between 0 and n , and with magnitude n , giving a derivation of length $O(m^{3m+2}n^2)$. For $\sigma = O\left(\sqrt[3]{\frac{\log n}{\log \log n}}\right)$, this is polynomial in n . \square

Theorem 8 (Constant coefficients). *Any semantic cutting planes proof of constant magnitude is polynomially simulated by a syntactic cutting planes proof of polynomial size and polynomial magnitude.*

Proof. We use the same construction as Theorem 7, but using the version of Theorem 6 which bounds magnitude. We now have a constant m number of variables, again bounded between 0 and n , and magnitude n . So we can simulate each semantic inference with length and magnitude polynomial in n . \square

Conclusions

We managed to efficiently simulate semantic proofs with very small coefficients using syntactic cutting planes, and we know that the simulation cannot be extended to exponentially large coefficients [FHL16]. The natural question left open is to check whether the simulation can be extended to semantic proofs with polynomial coefficients.

This is a proper proof system, because the inference rules can be verified in polynomial time. Hence it is even more compelling to understand whether syntactic CP can simulate it. Suppose we want to verify whether

$$a_1x_1 + \dots + a_nx_n = b$$

is satisfiable over $\bar{x} \in \{0, 1\}^n$, when a_i are integer numbers. Consider a branching program that queries x_1, \dots, x_n in turn and keeps track of the sum $a_1x_1 + \dots + a_nx_n$. Such a branching program has depth n and width $2 \sum |a_i| + 1$, since the partial sum is between $-\sum |a_i|$ and $\sum |a_i|$ at every step. Hence if the coefficients are small, we can check in polynomial time whether a value b is reachable by some choice of assignments. A simple extension of this procedure is sufficient to verify the soundness of any semantic CP inference with polynomial coefficients.

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