

# Short $\text{Res}^*(\text{polylog})$ refutations if and only if narrow Res refutations

(an answer to a question of Neil Thapen<sup>1</sup>)

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In this note we show that any  $k$ -CNF which can be refuted by a quasi-polynomial  $\text{Res}^*(\text{polylog})$  refutation has a “narrow” refutation in Res (i.e., of poly-logarithmic width). Notice that while  $\text{Res}^*(\text{polylog})$  is a complete proof system, this is not the case for Res if we ask for a narrow refutation. In particular is not even possible to express all CNFs with narrow clauses. But even for constant width CNF the former system is complete and the latter is not (see for example [BG01]). We are going to show that the formulas “left out” are the ones which require large  $\text{Res}^*(\text{polylog})$  refutations. We also show the converse implication: a narrow Resolution refutation can be simulated by a short  $\text{Res}^*(\text{polylog})$  refutation.

## References

The author does not claim priority on this result. The technical part of this note (Lemmata 1 and 2) bear similarity to the relation between  $d$ -depth Frege refutations and tree-like  $d+1$ -depth Frege refutations outlined in [Kra94]. Part of it had already been specialized to Res and  $\text{Res}(k)$  in [EGM04].

## Preliminaries

We consider a formula  $F$  in CNF form with  $m$  clauses and  $n$  variables. We fix  $k$  to be the width (i.e., number of literals) of the largest clause in  $F$ . The  $i$ -th clause of  $F$  is denoted as  $C_i$ , so that we can write  $F$  as  $\bigwedge_{i=1}^m C_i$ .

A *refutation* of size  $S$  for formula  $F$  is a sequence of formulas  $D_1, \dots, D_S$  such that  $D_i = C_i$  for  $1 \leq i \leq m$ , and any  $D_i$  for  $i > m$  is either an *axiom* or is *logically inferred* from two previous formulas in the sequence.

Different proof systems are characterized by the types formulas allowed as proof lines, by the axioms and by the logical inference rules. Further constraints on the structure of the refutation may be imposed: while this may cause refutations to be longer, it facilitates the process of searching for the refutation itself. A customary constraint is to impose *tree-like* structure on the refutation: with the exception of the axioms and the initial clauses of  $F$ , any formula in the sequence can be used at most once as premise for a logical inference. In this way the refutation looks like a binary tree in which every leaf is labelled either by an axiom or by an initial clause, and every internal vertex correspond to a formula inferred in the refutation. Notice that in a tree-like refutation any formula that is needed more than once must be re-derived from scratch. A proof system is called *dag-like* is no constraint on the structure of the proof is imposed (indeed the refutation looks like a directed acyclic graph).

**Res** : every line in the refutation is a clause (i.e., disjunction of literals), and there is a single inference rule:

$$\frac{A \vee x \quad B \vee \neg x}{A \vee B}$$

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<sup>1</sup>Apparently the present result can be proved by methods of Bounded Arithmetic. He asked whether there exists a simpler and more direct proof.

The *width* of a Res refutation is the maximum number of literals contained in a clause of the refutation. In this note we focus on refutations with poly-logarithmic width.

$\text{Res}^*(l)$ : the structure of the proof must be tree-like, every line in the refutation is a  $l$ -DNF, there is an axiom introduction rule:

$$\frac{}{\overline{l_1 l_2 \cdots l_s \vee \neg l_1 \vee \neg l_2 \vee \dots \vee \neg l_s}} \quad \text{for } 1 \leq s \leq l,$$

and an inference rule

$$\frac{A \vee l_1 l_2 \cdots l_s \quad B \vee \neg l_1 \vee \neg l_2 \vee \dots \vee \neg l_s}{A \vee B} \quad \text{for } 1 \leq s \leq l,$$

for any set of  $s$  literals  $l_1, l_2, \dots, l_s$ .

**Notes on the definition:** there are different ways to define  $\text{Res}^*(l)$ , which are all equivalent up to a polynomial size increase in the size of the refutation. Notice that there are neither weakening rule nor AND introduction rules: this allows to control how terms appear in the refutation and this makes the next proofs simpler. This is without loss of generality **since we are only dealing with refutations of CNFs**.

The cut rule is defined to require exactly the same number of literals on both sides. This is more rigid than usual, but **since the system is tree-like** this also is without loss of generality. This last restriction is not needed in the following proofs, but is kept to simplify notations.

## Main statement

In this note we prove that

**Theorem 1.** *Let  $F$  be a  $k$ -CNF.  $F$  has a quasi-polynomial size  $\text{Res}^*(\text{polylog})$  refutation if and only if has a Res refutation of poly-logarithmic width.*

Notice that any dag-like Res refutation of poly-logarithmic width has size at most quasi-polynomial.

## Proof of narrow Res simulation

The proof is based on the following Lemma, which immediately implies one direction of Theorem 1.

**Lemma 1.** *Let  $F$  be a  $k$ -CNF. If  $F$  has a refutation in  $\text{Res}^*(l)$  with  $L$  leaves, then has a Res refutation of width  $l \lceil \log L \rceil + \max\{k, l\}$ .*

*Proof.* The proof of the lemma is by induction on the number of leaves  $L$  in the refutation. If  $L = 1$  the initial CNF contains the empty clause, and the result is trivial.

Let us assume  $L > 1$ , and fix  $w = l \lceil \log L \rceil + \max\{k, l\}$ . Consider the last step of the refutation. Since it results in an empty DNF, it must be the result of a cut rule between formulas  $C = \bigwedge_{i=1}^s l_i$  (conjunction) and  $D = \bigvee_{i=1}^s \neg l_i$  (disjunction), for a set of  $s$  literals.

Since the refutation is tree-like the two proofs of  $C$  and  $D$  are disjoint, and the number of leaves in each proofs ( $L_C$  and  $L_D$  respectively) are such that  $L_C + L_D = L$ . Thus either  $L_C$  or  $L_D$  is less than or equal to  $\frac{L}{2}$ , and both are less than  $L$ . The proof is divided in two cases:

( $L_C \leq L/2$ ): Since  $F \vdash \bigwedge_{i=1}^s l_i$  in  $\text{Res}^*(l)$  with  $L_C$  leaves, by fixing  $l_i = 0$  we get that  $F|_{l_i=0} \vdash \square$  is proved by a  $\text{Res}^*(l)$  refutation which uses at most  $L_C$  leaves. By inductive

hypothesis the same refutation can be done in Res with width at most  $l\lceil\log L_C\rceil + \max\{k, l\} \leq l\lceil\log L\rceil + \max\{k, l\} - l \leq w - 1$ .

By weakening we have that  $F \vdash l_i$  in Res in width  $w$ , for any  $1 \leq i \leq s$ . Using such literals we can deduce  $F|_{l_1=1, \dots, l_s=1}$  from  $F$  in width  $k$  by removing any occurrences of literals  $\neg l_i$ . Since  $F \vdash \bigvee_{i=1}^s \neg l_i$  in  $\text{Res}^*(l)$ , we can prove  $F|_{l_1=1, \dots, l_s=1} \vdash \square$  in  $\text{Res}^*(l)$  with at most  $L_C < L$  leaves. By inductive hypothesis this refutation can be done in Res in width  $w$ . Composing the resolution proofs  $F \vdash l_i$  for  $1 \leq i \leq s$ , the proof of  $F, l_1, \dots, l_s \vdash F|_{l_1=1, \dots, l_s=1}$  and the proof  $F|_{l_1=1, \dots, l_s=1} \vdash \square$ , we get a Res refutation of width  $w$  of  $F \vdash \square$ .

( $L_D \leq L/2$ ): we may assume that  $s > 1$  because otherwise formulas  $C$  and  $D$  can be swapped and the reasoning for the previous case applies. Since  $F \vdash \bigvee_{i=1}^s \neg l_i$  in  $\text{Res}^*(\text{polylog})$  with  $L_D$  leaves we get that  $F|_{l_1=1, \dots, l_s=1} \vdash \square$ , with a  $\text{Res}^*(l)$  refutation with  $L_D \leq \frac{L}{2}$  leaves. By inductive hypothesis the same refutation can be done in Res with width at most  $l\lceil\log L_D\rceil + \max\{k, l\} \leq l\lceil\log L\rceil + \max\{k, l\} - l \leq w - l$ . By weakening Res proves  $F \vdash \bigvee_{i=1}^s \neg l_i$  in width  $w$ .

We now conclude arguing that Res proves  $F, \bigvee_{i=1}^s \neg l_i \vdash \square$  with width  $w$ . To see that observe the  $\text{Res}^*(l)$  proof of  $F \vdash \bigwedge_{i=1}^s l_i$ : each occurrence of  $\bigwedge_{i=1}^s l_i$  is introduced in such proof using the axiom  $\bigwedge_{i=1}^s l_i \vee \bigvee_{i=1}^s \neg l_i$ . Substitute such axiom with the new initial clause  $\bigvee_{i=1}^s \neg l_i$ . By an easy induction along the tree-like derivation, such transformation produces a  $\text{Res}^*(l)$  proof of  $F, \bigvee_{i=1}^s \neg l_i \vdash \square$  with  $L_C < L$  leaves. By inductive hypothesis this implies a Res refutation of width  $w$  (notice that the initial width of the formula increases, but that is accounted in the definition of  $w$ ). □

## Proof of the short $\text{Res}^*(l)$ simulation

The following lemma gives the other direction of Theorem 1.

**Lemma 2.** *Let  $F$  be any CNF. If  $F$  has a Res refutation of width  $w$  and size  $S$ , then  $F$  has a  $\text{Res}^*(w)$  refutation of size  $O(S)$ .*

*Proof.* Consider the Res refutation  $D_1, D_2, \dots, D_S$  of  $F$ . We define the sequence of  $w$ -DNFs  $E'_t = \bigvee_{i=1}^t \neg D_i$ . By backward induction on  $t$  from  $S - 1$  to  $0$  we are going to derive a  $w$ -DNF  $E_t$  such that the terms of  $E_t$  are a subset of the terms of  $E'_t$ . Since  $E'_0$  is the empty DNF that would conclude the proof.

For  $t = S - 1$  notice that  $E'_{S-1}$  contains  $x \vee \neg x$  for some variable  $x$  in  $F$ , which is an axiom in  $\text{Res}^*(w)$ .

Fix  $t < S - 1$  and consider  $D_a$  and  $D_b$  which has been used to derive  $D_{t+1}$ . For convenience write as follows

$$D_a \equiv A \vee x \quad D_b \equiv B \vee \neg x \quad D_{t+1} \equiv A \vee B \quad E_{t+1} \equiv \Delta \vee (\neg A \wedge \neg B)$$

for some  $w$ -DNF  $\Delta$ , some clauses  $A, B$  and some variable  $x$ . Terms of  $\Delta$  are all contained in  $E'_{t+1}$  by inductive hypothesis, and furthermore they are contained in  $E'_t$  because  $\neg D_{t+1}$  has been factored out. Employ the following tree-like deduction

$$(\neg A \wedge \neg x) \vee A \vee x \quad \text{Axiom of } \text{Res}^*(w) \quad (1)$$

$$(\neg B \wedge x) \vee B \vee \neg x \quad \text{Axiom of } \text{Res}^*(w) \quad (2)$$

$$(\neg A \wedge \neg x) \vee (\neg B \wedge x) \vee A \vee B \quad \text{Cut on term } x \text{ between (1) and (2)} \quad (3)$$

$$\Delta \vee (\neg A \wedge \neg B) \quad E_{t+1} \text{ deduced by induction hypothesis} \quad (4)$$

$$(\neg A \wedge \neg x) \vee (\neg B \wedge x) \vee \Delta \quad \text{Cut on term } \neg A \wedge \neg B \text{ between (3) and (4)} \quad (5)$$

Notice that formula (5) is a  $w$ -DNF, and its terms are contained in  $E'_t$ .

For  $t = 0$  we get the empty DNF. At each step  $E_t$  is derived in  $\text{Res}^*(w)$  using a single occurrence of formula  $E_{t+1}$ . That means that the whole refutation is tree-like and has  $O(S)$  proof lines.  $\square$

Notice that the  $w$ -DNFs have at most  $S$  terms each, so the size of the refutation has at most  $O(S^2)$  terms. For narrow Res refutations which require large clause space this bound is tight for our construction.

## References

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