Upper Bounds for Positional Paris-Harrington Games

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Abstract

We give upper bounds for a positional game — in the sense of Beck — based on the Paris-Harrington principle for bi-colorings of graphs and uniform hypergraphs of arbitrary dimension. The bounds show a striking difference with respect to the bounds of the combinatorial principle itself. Our results confirm a phenomenon already observed by Beck and others: the upper bounds for the game version of a combinatorial principle are drastically smaller than the upper bounds for the principle itself. In the case of Paris-Harrington games the difference is qualitatively very striking. For example, the bounds for the game on 3-uniform hypergraphs are a fixed stack of exponentials while the bounds on the corresponding combinatorial principle are known to be Ackermannian! For higher dimensions, the combinatorial Paris-Harrington numbers are known to be cofinal in the Schwichtenberg-Wainer Hiearchy of fast-growing functions up to ε_0, while we show that the game Paris-Harrington numbers are fixed stacks of exponentials.

Contents

1 Introduction and Motivation

We are motivated by the following phenomenon, observed in Combinatorial Game Theory (see, e.g., [1, 2, 3, 15]). When a combinatorial theorem (e.g., Ramsey Theorem, Van Der Waerden Theorem) is turned into a two-player positional game, then the witnessing functions for the game version are *drastically asymptotically slower* than the witnessing functions for the combinatorial principle. In this paper we present results showing that the same is the case for positional games based on the Paris-Harrington principle, in a very strong way.

The Paris-Harrington Theorem [11] for d-uniform hypergraphs (we henceforth use 'hypergraph' for 'uniform hypergraph' for the sake of brevity) says that, for every $k \geq d$, there exists a number N so large that any coloring of the complete d-hypergraph on vertex set $[k, N]$ admits a monochromatic sub-hypergraph on a set V of vertices with the following property: the cardinality of V is not smaller than the minimum element of V. The general version (with quantification on all dimensions $d \geq 2$) of this — seemingly innocent — variant of Ramsey Theorem is the most famous example of a natural mathematical finitary theorem that cannot be proved in strong theories like Peano Arithmetic, as shown by Harrington and Paris in [11]. For hypergraphs of fixed dimension the witnessing function for the Paris-Harrington principle grow extremely fast. E.g., for 3-hypergraphs, they grow as the Ackermann function, while for higher dimensions the bounds are cofinal in the Schwichtenberg-Wainer Hiearchy of fast-growin functions up to ε_0 (see [12]).

The work of Beck [1, 2, 3] has shown that in the case of Ramsey Theorem and of Van der Waerden Theorem the game functions are *drastically, qualitatively slower* than the witnessing functions of the combinatorial theorems. This has been confirmed by Nešetril and Valla [15] for some Ramsey classes of relational structures and seems to be a general phenomenon. Intuitively this is not surprising since in the game version the two players explore a small subtree of the tree of all colorings. We give a new example of this phenomenon by introducing Paris-Harrington positional games on graphs and hypergraphs of arbitrary dimension and proving upper bounds for the associated witnessing functions. We show that the difference between the game functions and the witnessing functions for the truth of the combinatorial principle are *dramatically* far apart. E.g., in the case of 3-hypergraphs, the game function is asymptotically bounded by a fixed stack of exponentials while the combinatorial principle has Ackermannian lower bounds. An analogous difference occurs for hypergraphs of dimensions larger than three.

Combinatorial games on graphs have interesting and deep connections with Automata Theory and Complexity Theory (parity games, reachability games, etc. See [10] for an overview). For example, the computational complexity of Ramsey games has been studied and proved to be **PSPACE**-complete by Slany [18]. We plan to investigate in future work these aspects of the games introduced in the present paper.

2 Ramsey and Paris-Harrington games

We introduce the combinatorial principles and the combinatorial games of interest for the present paper, along with the known corresponding bounds. We consider the following formulation of Ramsey Theorem for bi-colorings of hypergraphs.

Theorem 1. *(Ramsey Theorem, [17]) For every $d \geq 2$ for every $k \geq d$ there exists a number $r^d(k)$ which is the smallest number n such that for any edge-coloring of a complete d-hypergraph of n vertices in two colors, there is an homogeneous set of vertices of size k.*

In this paper we are mainly concerned with Paris-Harrington principles. The general Paris-Harrington Theorem (for arbitrary colorings of hypergraphs) was introduced in [11] as the first example of a mathematically natural witness of the incompleteness phenomenon for formal theories of arithmetic. The general version is known to be unprovable in first-order Peano Arithmetic [11], and the same holds if one restricts consideration to colorings with two colors [13].

In order to state the Paris-Harrington principle we need the following key definition. A set is called *relatively large* (or just *large*, for brevity) if its cardinality is not smaller than its minimum element. We denote the set of integers between a and b (included) by $[a, b]$. For a set S we denote the family of subsets of size d of S by $\binom{S}{d}$. The *Paris Harrington principle* is the following.

Theorem 2. *(Paris-Harrington Theorem [11]) For every $d \geq 2$, for every $k \geq d$, there exists a number $R^d(k)$ which is the smallest number n such that for any coloring of the elements of $\binom{[k,n]}{d}$ in two colors there exists a $H \subseteq [k, n]$ which is homogeneous and large.*

Obviously, the conclusion in the above principle is true for every number $n \geq R^d(k)$. Obviously $R^d(k) \geq r^d(k) + k - 1$ always holds.

We briefly recall the known bounds for the Ramsey and Paris-Harrington principles of interest for the present paper.

Diagonal Ramsey numbers (in our notation $r^d(k)$) are an active topic of research, rich in difficult open problems. For the sake of this paper, it is enough to recall the following classical bounds. For graphs the following bounds are due to Erdős [6].

$$2^{k/2} < r^2(k) \leq 2^{2k}.$$

The currently best upper bound is due to Conlon [4]. For $d \geq 3$ the following bounds are known [7]:

$$2^{k^2/6} < r^3(k) < 2^{2^{4n}}, \text{ and } 2^{2^{k^2/6}} < r^4(k) < 2^{2^{2^{4k}}}.$$

Let $\mathsf{tower}_d(k)$ denote the d-th iteration of the exponential function in base 2, i.e., $\mathsf{tower}_0(k) = k$ and $\mathsf{tower}_{d+1}(k) = 2^{\mathsf{tower}_d(k)}$. In general, the following holds.

$$\mathsf{tower}_{d-2}(k^2/6) < r^d(k) < \mathsf{tower}_{d-1}(4k). \tag{1}$$

The currently best bounds are due to Conlon, Fox, and Sudakov [5].

Now we recall the known bounds on the Paris-Harrington principle. For bi-colorings of graphs the Paris-Harrington principle is only slightly stronger than Ramsey Theorem and is known to have double exponential upper bounds. The following bounds have been established by Erdős and Mills [9, 14]. There exist constants $\alpha, \beta, N > 0$ such that for all $k \geq N$

$$k^{2^{\alpha k}} < R(k) < k^{2^{2^{\beta k}}} \quad . \tag{2}$$

As observed in [9], the results on the asymptotics of the Paris-Harrington numbers for colorings of graphs in $c > 2$ colors imply that $R^3(k)$ grows essentially as fast as Ackermann's function. In particular it has no primitive recursive upper bound.

To describe the bounds on Paris-Harrington numbers for $d > 3$ one needs to introduce a hierarchy of fast-growing functions indexed by notations for ordinals below ε_0 (the smallest fix-point of the function $\alpha \mapsto \omega^\alpha$). The hierarchy is called the Schwichtenberg-Wainer Hierarchy and commonly denoted by $(F_\alpha)_{\alpha<\varepsilon_0}$. Since we do not need the details of this hierarchy in the present paper, we skip them and refer the reader to, e.g., [12]. For the sake of comparison with our results, the following highlights are sufficient. The backbone of the hierarchy is constituted by the functions F_{ω_d}, where $\omega_1 = \omega$ and $\omega_{d+1} = \omega^{\omega_d}$. First, F_ω is a variant of the Ackermann function. In general, the following fundamental result from Proof Theory gives a hint of the growth-rate of the functions of the hierarchy (see [19] for a classical textbook treatment). For $d \geq 1$, the function F_{ω_d} eventually dominates the (computable) functions that have a Σ_1-definition and a proof of termination in the system $I\Sigma_d$ obtained from Peano Arithmetic by restricting the induction axiom to Σ_d-formulas. On the other hand, we know that F_{ω_d} is a lower bound on the witnessing function for the Paris-Harrington principle for colorings of $d+2$ hypergraphs in two colors (see [12]). Thus, the witnessing functions for the Paris-Harrington principle are extremely fast-growing for dimensions larger than two.

We now discuss how Ramsey and Paris-Harrington theorems are naturally turned into positional two-players games.

Beck [1] defined the following game version of Ramsey Theorem for hypergraphs of dimension $d \geq 2$. The *Ramsey* (d,k)*-game* (denoted by $RAM(d,k,N)$) with *dimension* d and *target* k is played on a board consisting in the complete d-hypergraph on N vertices. The two players, Maker (female) and Breaker (male), take moves in turn, starting with Maker. Each player at each move can color a hyperedge in his/her own color. The goal of Maker is to color a complete sub-hypergraph of size k in her own color. The goal of Breaker is to avoid that this happens.

A simple (but non-constructive) *strategy stealing* argument shows that Maker has a winning strategy if N is larger than the Ramsey number $r^2(k)$ (see, e.g., [2]). Let $\hat{r}^d(k)$ denote the smallest number such that Maker has a winning strategy in game $RAM(d,k,\hat{r}^d(k))$. We call $\hat{r}^d(k)$ the *game number* for the Ramsey (d,k)-game. In his fundamental studies on positional games Beck [1, 2, 3] has proved the following upper bounds on the game numbers for Ramsey games of arbitrary dimension.

Theorem 3. *(Beck [2], Pekeč [16])*

1. *If* $N \leq 2^{k/2}$ *then Breaker has a winning strategy for the game* $RAM(2,k,N)$*,*
2. *if* $N \geq 2^{k+2}$ *then Maker has a winning strategy for the game* $RAM(2,k,N)$*.*

Theorem 4. *(Beck, [2]) For every* $d \geq 3$ *there are positive constants* c_d *and* c'_d *such that*

1. *If* $N \leq 2^{c_d \cdot k^{d-1}}$ *then Breaker has a winning strategy in the game* $RAM(d,k,N)$*, and*
2. *if* $N \geq 2^{c'_d \cdot k^d}$ *then Maker has a winning strategy in the game* $RAM(d,k,N)$*.*

The Paris-Harrington Theorem is naturally turned into a positional combinatorial two-players game in a similar way. The *Paris-Harrington* (d,k)*-game* (denoted by $PH(k,N)$) is played on a board consisting of a complete hypergraph with vertex set $[k,N]$. Maker and Breaker are as in the Ramsey Game. The goal of Maker is now to color a complete relatively large sub-hypergraph in her own color.

Let $\hat{R}^d(k)$ denote the smallest number such that Maker has a winning strategy in game $PH(d,k,\hat{R}^d(k))$. We call $\hat{R}^d(k)$ the *game number* for the Paris-Harrington (d,k)-game.

As in the case of Ramsey games, a standard *strategy stealing* argument shows that Maker has a winning strategy in $PH(d,k,N)$ if N is larger than the Paris-Harrington number $R^d(k)$.

3 Time to meet the Maker

We prove upper bounds for the Paris-Harrington (d,k)-games for every $d \geq 2$ and $k \geq d$. We first discuss the case of graphs and then the case of hypergraphs of dimension $d \geq 3$.

3.1 The case of graphs

We consider the Paris-Harrington $(2,k)$-game, for $k \geq 2$. Using a suitable (greedy-type) strategy for the Maker we reduce the problem to the problem of lower bounding the largest independent set in a graph on n vertices and n edges. We first discuss the latter problem.

It is well known that a simple graph with degree at most d is $(d+1)$-colorable in the following way: color all vertices one by one in an arbitrary order, choosing at each step a color which does not appear in the vertex's neighborhood. Each vertex has at most d neighbors, so $d+1$ colors are always sufficient. We want to discuss a similar result for graphs of *average* degree $d \geq 0$. Notice that such graphs could have polynomial size cliques even if d is constant, thus a small coloring is impossible. We decide then to relax the request. A $(d+1)$-colorable graph with n vertices has a independent set of size at least $\frac{n}{d+1}$ (hint: pick the largest color class). We show that such a big independent set can be guaranteed even if d is just the average degree.

Lemma 1. *Any graph G with n vertices and average degree d has a independent set of size $\frac{n}{d+1}$.*

Proof. We compute an independent set by the following randomized greedy procedure. We sort the vertices in a random order. We start with an empty S, and we scan all vertices according to the chosen order. Each time we meet a vertex with no neighbors in S we put it in S, and we carry on with the scan. It is clear that S is a independent set. Let us define the random variable X_v which is 1 if $v \in S$ and 0 otherwise. Then

$$|S| = \sum_{v \in V(G)} X_v.$$

We denote the degree of a vertex v as d_v. We now focus on the expected value $\mathbb{E}X_v$. A vertex v is always chosen to be in S if it occurs before its neighbors in the random order. Then $\mathbb{E}X_v \geq \frac{1}{d_v+1}$. Thus we get

$$\mathbb{E}|S| \geq \sum_{v \in V(G)} \frac{1}{d_v+1} \geq \frac{n^2}{\sum_{v \in V(G)} (d_v+1)} = \frac{n}{d+1},$$

where the first inequality comes from the linearity of expectation and the second inequality comes by setting $p_i = \frac{1}{n}$ in the following formulation of the Harmonic Mean-Arithmetic Mean inequality.

Proposition 1. *(Harmonic Mean-Arithmetic Mean inequality) For non negative $p_1 \dots p_n$ such that $p_1 + \cdots + p_n = 1$ we get that*

$$\frac{1}{\frac{p_1}{x_1} + \frac{p_2}{x_2} + \cdots + \frac{p_n}{x_n}} \leq p_1 x_1 + p_2 x_2 + \cdots + p_n x_n.$$

Since the expected value of S has size at least $\frac{n}{d+1}$ there exists an ordering of the vertices for which an independent set of that size is achieved. □

Corollary 1. *A graph with n vertices and n edges has a independent set of size at least $\lceil \frac{n}{3} \rceil$.*

Proof. Since there are n edges, it follows that the average degree is two. The statement follows from the previous theorem and from the integrality of the set size. □

Note that the previous statement is tight for any n: if $n = 3m + r$, consider a graph made of m triangles, with a path of r edges joined to an arbitrary vertex in an arbitrary triangle. This graph contains exactly n edges. If $r = 0$ then the largest independent sets have size m, otherwise they have size $m + 1$.

Theorem 5. *If $N \geq k - 4 + 6 \cdot \hat{r}^2(k-1)$, then Maker has a winning strategy in the game $PH(2, k, N)$.*

Proof. Without loss of generality we assume $N = k - 4 + 6\hat{r}^2(k-1)$. The Maker strategy is divided in two phases. In the first one she greedily connects $3 \cdot \hat{r}^2(k-1) - 2$ vertices in $[k+1, N]$ to vertex k: her first move is $\{k, k+1\}$, and at each subsequent step Maker picks the smallest v such that $\{k, v\}$ has not been yet taken by the Breaker. The value of N ensures that Maker can play this move at least $3 \cdot \hat{r}^2(k-1) - 2$ times. After that many rounds, Maker has colored $3 \cdot \hat{r}^2(k-1) - 2$ edges of the form (k, v) for $v \in [k+1, N]$.

Let us call U such a set of vertices. The game has lasted $3 \cdot \hat{r}^2(k-1) - 2$ rounds so far, thus at most $3 \cdot \hat{r}^2(k-1) - 2$ edges among vertices of U have been captured by Breaker (we remark that the last move of the previous phase was made by Breaker). Corollary 1 implies that there exists $W \subseteq U$ of size

$$\left\lceil \frac{U}{3} \right\rceil \geq \left\lceil \frac{3 \cdot \hat{r}^2(k-1) - 2}{3} \right\rceil = \hat{r}^2(k-1),$$

such that no edges touching W have been captured by neither players.

The next move is for Maker, who now starts the second phase of the strategy. From now on Maker plays a strategy for winning the game $RAM(2, k, |W|)$ on the set of vertices W. Such a strategy exists since $|W| \geq \hat{r}^2(k-1)$, and since no pair in W has been touched so far.

At the end of the second phase Maker wins the game on W, and thus has captured a clique $C \subseteq W$ of size $k-1$. All vertices in W have been connected to k by Maker because of the first phase of the strategy: it follows that $C \cup \{k\}$ is a relatively large clique since it has size k and its minimum element is k. □

From the bounds in Assertion (2) of Theorem 3 we get the following Corollary.

Corollary 2. *For all $k \geq 2$,*

$$\hat{R}^2(k) \leq (12 + o(1)) \cdot 2^k.$$

Note that the board size proved to be sufficient for Maker to win is drastically smaller than the known lower bounds on $R^2(k)$, which are double exponential in k, see Equation 2.

3.2 The general case

We prove an upper bound on the game function for the Paris-Harrington game of dimension $d \geq 3$. The construction is uniform in the dimension. Again we reduce the Paris-Harrington game to a standard Ramsey Game.

Theorem 6. *For all $d \geq 3$,*

$$\hat{R}^d(k) \leq \hat{r}^d(r^{d-1}(k)) + k + 1.$$

Proof. We describe a winning strategy for the Maker on the board $[k, N]$ where $N = \hat{r}^d(r^{d-1}(k)) + k + 1$.

Consider the interval $I = [k+2, N]$. Maker has a strategy for winning the game $RAM(d, r^{d-1}(k), |I|)$, because $|I| = \hat{r}^d(r^{d-1}(k))$. Maker starts by playing this strategy on the set I, and indeed her first move of the game is the first move of that strategy.

Simultaneously Maker keeps the following behavior: every time Breaker picks an hyperedge of the form $\{k\} \cup X$ where $X \in \binom{I}{d-1}$, Maker answers by picking the hyperedge $\{k+1\} \cup X$; dually if Breaker picks $\{k+1\} \cup X$, then Maker picks $\{k\} \cup X$ at the next move. If Breaker picks an edges of the form $\{k, k+1\} \cup Y$ for some Y, Maker makes an arbitrary move: the intuition is that such Breaker moves are useless against Maker's strategy. While Maker could take advantage of such moves in order to obtain an earlier win, we choose to waste Maker's move in this case in order to simplify the following analysis.

The relevant moves by both players are either hyperedges in $\binom{I}{d}$ or hyperedges in $\{k, k+1\} \times \binom{I}{d-1}$. The way the two types of moves interleave is such that Maker is always one move ahead of Breaker when it comes to the game played on I. Maker is playing a winning strategy for the Ramsey $(d, r^{d-1}(k))$-game there, so she eventually captures all d-hyperedges on a vertex set $U \subseteq I$ of size $r^{d-1}(k)$.

From now on we consider the subsets $X \subseteq U$ of size $d-1$. Thanks to the moves of the "second type" it holds that — after each Maker's move — either both $\{k\} \cup X$ and $\{k+1\} \cup X$ have been picked — one by Maker and the other by Breaker — or none of them has been picked by either Maker or Breaker. The moves of the play of the "second type" determine a partial coloring of $\binom{U}{d-1}$ as follows: if $\{k\} \cup X$ belongs to Maker then X is colored k, otherwise X is colored $k+1$ (note that in the latter case the hyperedge $\{k+1\} \cup X$ belongs to Maker).

At some point $\binom{U}{d-1}$ will be completely colored (the board is finite, so the Breaker cannot drag his feets indefinitely). Since U has size $r^{d-1}(k)$, there exists $U' \subseteq U$ such that U' has size k and is monochromatic for the coloring induced by the moves of "second type". If the color of U' is k then $\{k\} \cup U'$ is a homogeneous relatively large set and thus a winning set for the Maker. If the color of U' is $k+1$ then $\{k+1\} \cup U'$ is such a winning set for the Maker. □

Corollary 3. *For every $d \geq 3$, there exists a constant c'_d such that for every $k \geq d$, if*

$$N \geq 2^{c'_d \cdot \mathsf{tower}_{d-2}(4k)^d} + k + 1,$$

then Maker has a winning strategy for the Paris-Harrington game $PH(d, k, N)$.

Proof. The following inequalities hold, where the constant c'_d is the same as in Assertion (2) of Theorem 4.

$$\begin{aligned}\hat{R}^d(k) &\leq \hat{r}^d(r^{d-1}(k)) + k + 1 \\ &\leq 2^{c'_d \cdot (r^{d-1}(k))^d} + k + 1 \\ &\leq 2^{c'_d \cdot \mathsf{tower}_{d-2}(4k)^d} + k + 1\end{aligned}$$

The first inequality holds by Theorem 6, the second inequality holds by Beck's upper bound on the Ramsey game (Assertion 2 of Theorem 4), and the third inequality holds by the known bounds on Ramsey numbers, see Equation 1. □

The upper bounds for the Paris-Harrington (d, k)-game in the previous Corollary should be compared with the lower bounds on the corresponding Paris-Harrington principles. For example, for $d = 3$, Corollary 3 gives an upper bound of the order of $2^{c \cdot 2^{(d4k)}} + k + 1$, while the Paris-Harrington numbers for 3-hypergraphs dominate the primitive recursive functions! Indeed, our bounds on the game functions are primitive recursive (even elementary) for each $d \geq 2$ while for $d \geq 3$ the Paris-Harrington numbers are cofinal in the fast-growing hierarchy $(F_\alpha)_{\alpha<\varepsilon_0}$.

4 Conclusion

We have introduced positional games in the sense of Beck [3], based on the Paris-Harrington principle [11] for bi-colorings of hypergraphs of arbitrary dimension. We have confirmed the following general pattern (see [1, 2, 15]) the upper bounds on the positional games are *dramatically smaller* than the lower bounds on the corresponding combinatorial principles. For Paris-Harrington games the difference is particularly striking. For example, while the Paris-Harrington Theorem for bi-colorings of hypergraphs starts to have no primitive recursive upper bound from dimension three on, the bounds on the corresponding positional game are fixed stacks of exponentials. We plan to consider the following natural problems in future research: (1) compute lower bounds for the Paris-Harrington games, (2) study the strong version of the games, in which the Breaker wins if he colors a large complete sub-hypergraph in his own color, and (3) study the computational complexity of Paris-Harrington games (as done in [18] for graph Ramsey games).

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